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# Note on Forced Burgers Turbulence

Robert H. Kraichnan*
*PMB 108, 369 Montezuma, Santa Fe, NM 87501-2626*

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A putative powerlaw range of the probability density of velocity gradient in high-Reynolds-number forced Burgers turbulence is studied. In the absence of information about shock locations, elementary conservation and stationarity relations imply that the exponent $-\alpha$ in this range satisfies $\alpha \geq 3$, if dissipation within the powerlaw range is due to isolated shocks. A generalized model of shock birth and growth implies $\alpha = 7/2$ if initial data and forcing are spatially homogeneous and obey Gaussian statistics. Arbitrary values $\alpha \geq 3$ can be realized by suitably constructed homogeneous, non-Gaussian initial data and forcing.



Burgers equation was originally proposed as a simplified dynamical system that might point at statistical procedures applicable to Navier-Stokes turbulence. That goal seems still far away. Meanwhile, a body of research has been devoted to Burgers turbulence itself. The particular topic addressed in the present paper is the power law of a part of a probability density for Burgers turbulence forced at high Reynolds number. The conclusion reached is that determination of the correct power law requires detailed examination of the dynamical behavior of explicit structures, namely shocks. This perhaps has discouraging implications for any general theory of the higher statistics of Navier-Stokes turbulence, where more varied and plastic flow structures must be confronted.

There has been considerable interest in a putative powerlaw range of the probability density $Q(\xi)$ of velocity gradient $\xi = u_x$ for high-Reynolds-number Burgers turbulence forced at large scales. This range, of form $Q(\xi) \propto |\xi|^{-\alpha}$, is believed to occupy negative $\xi$ values intermediate between those near the central peak of $Q$ and those characteristic of the shock interiors. Proposals for $\alpha$ include 2 [1,2], the range 5/2 to 3 [3,4], 3 [5], and 7/2 [6–9].

It should be emphasized at the outset that the high-Reynolds-number limit is not the only case of theoretical interest. As with Navier-Stokes turbulence, the construction of faithful analytical approximations at finite Reynolds numbers remains a challenge.

E and Vanden Eijnden [7–9] advance and clarify the mathematics of the infinite-Reynolds-number limit and provide some valuable tools. One is a simple representation of the effects of shock interactions on $Q$ in terms of the rate at which fluid is swallowed by the shocks. Another is a steady-state integral representation of the asymptotic large-$|\xi|$ form of $Q(\xi)$ in terms of the dissipation term $F(\xi)$ in the $Q$ equation of motion. Another is an explicit shock-birth model that implies $\alpha = 7/2$.

The present paper explores the relation between the form of $F(\xi)$ and the value of $\alpha$. If viscous effects in the $-\alpha$ range are due to isolated shocks, the form of $F(\xi)$ in this range expresses the relative likelihood, weighted by shock strength, that a shock occurs in a fluid environment with given $\xi$.

In the absence of an explicit shock-growth model, or other source of information about the distribution of shocks, the integral equation for $Q$ is found to yield $\alpha \geq 3$. The slightly stronger bound $\alpha > 3$ is stated in [7] and [8], but with the recognition that $\alpha = 3$ may be realized under particular circumstances. The limit of infinite Reynolds number is taken in the present paper without making a split [7] of the velocity field into shock interiors and external field. At the end, the analysis is extended to the split-field representation.

A more general model of shock growth is presented here. It is independent of details of internal shock structure. In this model, one examines the length of time during which $\xi$ can steepen within a fluid element before the fluid element hits a shock. The model implies $\alpha = 7/2$ if the forcing is statistically homogeneous and Gaussian. More general statistically homogeneous forcing can realize arbitrary values $\alpha \geq 3$.

Let $R = u_0 L/\nu$, $\xi_0 = u_0/L$, $\xi_S = R\xi_0$, where $u_0$, $L$, $\nu$ are root-mean-square velocity, spatial macroscale, and viscosity. The order of magnitude of gradients within typical shocks is $\xi_S$. The forced Burgers equation is

$$u_t + u_x u = \nu u_{xx} + f, \tag{1}$$

where $f$ is the (statistically stationary) forcing field. If $f$ has infinitely short correlation times, (1) leads to

$$Q_t = \xi Q + (\xi^2 Q)_\xi + BQ_{\xi\xi} + F, \tag{2}$$

where

$$F(\xi, t) = -\nu(H(\xi, t)Q(\xi, t))_\xi, \quad H(\xi, t) = \langle \xi_{xx} | \xi \rangle, \tag{3}$$

and the parameter $B$ measures the strength of forcing of $\xi$ [5]. The first term on the right side of (2) represents loss or gain of measure due to squeezing or stretching of the fluid, the second term describes relaxation of positive $\xi$ and steepening of negative $\xi$, and $F$ includes all viscous effects.

Statistical homogeneity requires $\langle\xi\rangle = \int_{-\infty}^{\infty} \xi Q d\xi = 0$. Multiplication of (2) by $\xi$ and integration over all $\xi$ shows that this condition is preserved, provided that $Q$ vanishes strongly enough at $\pm\infty$. The result depends on

$$\int_{-\infty}^{\infty} \xi F(\xi) d\xi = 0, \tag{4}$$

which follows from (3) and $\langle\xi_{xx}\rangle = 0$.

If $f$ has spectral support effectively confined to wavenumbers $O(1/L)$, then $\xi_0 = O(B^{1/3})$. In this case, it is widely agreed that the steady-state $Q$ has a complicated form in the limit $R \to \infty$. There is a central peak of width $O(\xi_0)$ whose form is $R$-independent in the limit. There is faster-than-algebraic decay as $|\xi| \to \infty$. For $\xi > 0$, this decay has the specific form $\exp(-\xi^3/3B)$, with an algebraic prefactor. The rapidly-decaying tail for $\xi < 0$ (far-left tail) includes $|\xi| \geq O(\xi_S)$. It is preceded, at smaller $|\xi|$, by an algebraic tail of form $1/R|\xi|$ ($-1$ range) associated with the shoulders of developed shocks. Between the $-1$ range and the central peak, an inner algebraic tail of form $Q(\xi) \propto \xi_0^{\alpha-1}|\xi|^{-\alpha}$ is expected. This tail is driven by the inviscid steepening of negative gradients. Proposals for the value of $\alpha$ have ranged from 2 to 7/2.

The $-\alpha$ range is infinite if $R$ is, but is confined to $|\xi|$ smaller than the $O(\xi_S)$ gradients inside the shocks. Thus the range is restricted to $\xi > -\xi_M \equiv -R^z\xi_0$, with $z \leq 1$. In fact, both shock analysis and simulation show that the $-\alpha$ range is masked by the $-1$ range at negative enough $\xi$. The masking further restricts the observable $-\alpha$ range to $z = 1/(\alpha - 1)$, a result that follows from setting $1/R\xi_M = \xi_0^{\alpha-1}\xi_M^{-\alpha}$. It should be emphasized that the transition from $Q(\xi) \propto |\xi|^{-\alpha}$ to $Q(\xi) \propto 1/R|\xi|$ at $\xi = O(-\xi_M)$ is a masking, not a dynamical transition. The $-1$ range is associated with the shoulders of quasi-stationary mature shocks while the $-\alpha$ range is associated with inviscid steepening of gradients away from shocks. The latter process continues, for at least some fluid elements, up to $|\xi| = O(\xi_S)$.

The behavior of $Q$ in all ranges is linked by (2) to the form of $F(\xi)$. For large negative $\xi$, the relationship is expressed in especially clear form by the integral representation

$$Q(\xi) \approx |\xi|^{-3} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' \quad (-\xi \gg \xi_0) \tag{5}$$

derived by E and Vanden Eijnden [7]. An alternative form of (5) is [5]

$$\nu H(\xi) \approx \xi^2 + \frac{1}{Q(\xi)} \int_{-\infty}^{\xi} Q(\xi')\xi' d\xi' \quad (-\xi \gg \xi_0). \tag{6}$$

or by (3),

$$F \approx -\xi Q - (\xi^2 Q)_\xi \quad (-\xi \gg \xi_0). \tag{7}$$

Equation (7) says that in a steady state, for $\xi$ such that the $B$ term is negligible, the $F$ term in (2) must balance both the gradient steepening term $(\xi^2 Q)_\xi$ and the loss-of-measure term $\xi Q$. Equations (5)–(7) do not require large $R$.

According to (5), the behavior of $Q$ in the $-\alpha$ range depends on, first, the value $\int_{-\infty}^{-\xi_M} \xi' F(\xi') d\xi'$ of the integral at the outer end of the range; second, the $\xi$ dependence of $F(\xi)$ within the $-\alpha$ range, which determines the growth of the integral within the range. If $F(\xi)$ in the $-\alpha$ range arises solely from isolated shocks, the analysis of [7] gives an immediate bound on $\alpha$. This analysis expresses a simple physics: If a shock lies in surrounding fluid with gradient $\xi$, then this fluid is swallowed by the convergence at the shock, thereby decreasing $Q(\xi)$. This means that $F(\xi)$ cannot be positive in the $-\alpha$ range. Since $\xi < 0$, the integral in (5) therefore cannot increase as $|\xi|$ decreases within the range, and $\alpha < 3$ is impossible.

The cases $\alpha = 3$ and $\alpha > 3$ imply qualitatively different magnitudes in (5). If $\alpha = 3$, then (5) immediately gives $\int_{-\infty}^{-\xi_M} \xi' F(\xi') d\xi' = O(\xi_0^2)$. It is then required that $F$ be small enough that $\int_{-\infty}^{\xi} \xi' F(\xi') d\xi' = O(\xi_0^2)$ if $\xi$ lies anywhere within the $-\alpha$ range. If $\alpha > 3$, (5) immediately gives $\int_{-\infty}^{-\xi_M} \xi' F(\xi') d\xi' \to 0$ as $\xi_M/\xi_0 \to \infty$. Then $F(\xi) \propto -|\xi|^{1-\alpha}$ within the range will give $Q(\xi) \propto |\xi|^{-\alpha}$. This means $F(\xi) \propto \xi Q(\xi)$.

If dissipation within the $-\alpha$ range is due to isolated shocks, the physical difference between $\alpha = 3$ and $\alpha > 3$ lies in the weighted relative likelihood that shocks exist in environments with given $\xi$. $F(\xi) \propto -Q(\xi)$, a case of Polyakov's closure [3,4], is one form that is consistent with $\alpha = 3$. This corresponds to unbiased placement of shocks in the fluid [10]. $F(\xi) \propto \xi Q(\xi)$ represents bias toward larger $|\xi|$ within the $-\alpha$ range. Since $Q(\xi)$ falls off rapidly in the $-\alpha$ range, even this form puts most of the shocks in fluid with $|\xi| = O(\xi_0)$.

If $\alpha = 3$, the inviscid steepening of gradient in most fluid elements with negative $\xi$ continues until $|\xi| = O(\xi_S)$. If the dissipative process that stops further steepening is absorption by well-defined shocks, these shocks must occur with sufficient weight in environments with $|\xi| = O(\xi_S)$. Since $\xi_S$ measures typical gradients within shocks, other processes, such as new shock formation, may actually dominate the dissipation.

Further insight concerning the relation between $F$ and $\alpha$ is provided by the equation of motion (2) for $Q$ and, in particular, by the steady-state equation $Q_t = 0$. A variety of global assumed forms for $F$, including $F(\xi) \propto -Q(\xi)$, can be adjusted, by tuning of the constant of proportionality, to give a steady-state $Q(\xi)$ that vanishes strongly as $\xi \to +\infty$, is positive everywhere, and has a range $\alpha = 3$. In order to be physically relevant, $F(\xi)$ for negative $\xi$ beyond $\xi_M$ must be shaped to give consistent $1/R|\xi|$ and far-tail ranges for $Q$. Global model forms can also be constructed that yield a range with $\alpha > 3$.

The quantity $\langle\xi\rangle$ is identically conserved at value zero. It is of interest to examine the way in which flow of $\langle\xi\rangle$ through the $-\alpha$ range depends on the value of $\alpha$. Let (2) be multiplied by $\xi$ and integrated over the range $(-\xi_M, \infty)$ to yield

$$\int_{-\xi_M}^{\infty} \xi Q_t(\xi) d\xi = \xi_M^3 Q(-\xi_M) + \int_{-\xi_M}^{\infty} \xi F(\xi) d\xi. \quad (8)$$

In writing (8), a partial integration is performed, it is assumed that $Q$ vanishes strongly enough at $+\infty$, and it is assumed that $\xi_M/\xi_0$ is large enough that the $B$ term is negligible.

The left side of (8) is the rate of increase of $\langle\xi\rangle$ in the range $(-\xi_M, \infty)$. On the right side, $\xi_M^3 Q(-\xi_M)$ is the rate of increase due to flow of negative contribution to $\langle\xi\rangle$, through the boundary at $-\xi_M$, to the range $(-\infty, -\xi_M)$. This flow is due to inviscid steepening of gradients. The $F$ term on the right side of (8) is the rate of increase of $\langle\xi\rangle$, or decrease of $\langle-\xi\rangle$, in the range $(-\xi_M, \infty)$, due to viscous interaction with isolated shocks and any other dissipative structures that may be present.

The nature of the $F$ term in (8) must be understood clearly. The present analysis is at finite $R$, with the eventual limit $R \to \infty$ considered. There is no sudden jump of $\langle\xi\rangle$ contribution from $-\alpha$ range to shock interiors. What does happen at large $R$ is that a fluid element with $\xi$ in the $-\alpha$ range hits the shock and then suffers a very rapid, but continuous, steepening until its $\xi$ is the order of that in the shock interior. The sum of these events in the entire range is expressed by the $F$ term in (8).

In a steady state, the right side of (8) vanishes. The implications differ in the cases $\alpha = 3$ and $\alpha > 3$. If $\alpha > 3$, the boundary-flow term in (8) vanishes in the limit $\xi_M/\xi_0 \to \infty$. This means that the $F$ integral term vanishes also in the limit. In view of (4), where the limits are true infinity (infinite compared to $\xi_S$), it follows that $\int_{-\infty}^{-\xi_M} \xi F(\xi) d\xi$ also vanishes in the limit. This does not mean that $F(\xi)$ tends to zero in the limit for all $\xi$ in the range $(-\infty, -\xi_M)$. Instead, there are both positive and negative contributions that cancel in the limit. In general, $F(\xi)$ is positive in the $-1$ range, as illustrated by (9) and (10) below.

If $\alpha = 3$, the boundary term in (8) tends to a nonzero positive constant in the limit. A steady state then implies that the $F$ integral in (8) is negative and that $\int_{-\infty}^{-\xi_M} \xi F(\xi) d\xi$ is positive. This means that the contributions from the negative and positive regions of $F$ in the range $(-\infty, -\xi_M)$ do not cancel. The nonzero value of $\int_{-\infty}^{-\xi_M} \xi F(\xi) d\xi$ needed in the case $\alpha = 3$ has already been noted in the discussion of (5).

For both $\alpha > 3$ and $\alpha = 3$, the boundary flow through an arbitrary point $-\xi_\alpha$ fully within the $-\alpha$ range is independent of $\xi_\alpha$ in the limit $R \to \infty$. Thus if $\xi_M$ in (8) is replaced by any $\xi_\alpha$ such that $\xi_\alpha/\xi_M \to 0$, $\xi_\alpha/\xi_0 \to \infty$ in the limit, then the limiting value of the boundary flow vanishes if $\alpha > 3$. The flow has a value $O(\xi_0^2)$, independent of $\xi_\alpha$, if $\alpha = 3$.

The form of $Q$ and $F$ in the far-left tail and $-1$ range actually is not constrained by whether $\int_{-\infty}^{-\xi_M} \xi F(\xi) d\xi$ vanishes or is $O(\xi_0^2)$ as $R \to \infty$. The key is the value of $z = 1/(\alpha - 1)$, which gives the position of the join between $-1$ and $-\alpha$ ranges at $-\xi_M = -R^z \xi_0$. Consider the generic form

$$Q(\xi) \approx Z(\xi/R\xi_0)/R|\xi| \quad (-\xi \gg \xi_0) \quad (9)$$

in the far-tail and $-1$ ranges, where $Z$ vanishes strongly as $\xi \to -\infty$, $Z \to C_z$ as $\xi/R\xi_0 \to -0$, and $C_z$ is an $O(1)$ constant. If this form is substituted into (7), a consequence is

$$\int_{-\infty}^{-\xi_M} \xi F(\xi) d\xi \approx C_z R^\beta \xi_0^2 \quad (R \gg 1), \quad (10)$$

where $\beta = (3 - \alpha)/(\alpha - 1)$. If $\alpha = 3$, this gives the needed $O(\xi_0^2)$ result. If $\alpha > 3$, the right side vanishes at $R = \infty$. Thus the value of the boundary flow automatically adjusts to the value of $\alpha$ that is determined by the form of $F$ in the $-\alpha$ range. In other words, it adjusts to the probability distribution of shock occurence in the $-\alpha$ range.

Explicit models of shock development lead to explicit forms of $F(\xi)$. The following model generalizes those of [6–9]. It assumes large $R$ but does not invoke the internal shock structure. The development is followed before and after shock birth. Take the unforced case first. Consider an initial velocity field of form

$$u(x, 0) \approx \xi_0(-ax + bx|x/L|^p) \quad (p > 0) \quad (11)$$

in the vicinity of a point $x = 0$ where a shock will form at time $1/a\xi_0$. Here $a$ and $b$ are $O(1)$ positive constants. If $p$ is an even integer, all $x$ derivatives of $u$ exist at $x = 0$. If $p$ is an odd integer or non-integer, only the derivatives of order $n \le p + 1$ exist.

This form of initial velocity field leads to $\alpha = 3 + 1/p$, a result that can be verified in several ways. The following is a simple qualitative argument. The inviscid evolution of velocity gradient in a Lagrangian frame satisfies $\dot{\xi} = -\xi^2$. It then follows from (11) that the time of initial shock formation is $t_0 = 1/a\xi_0$. The negative gradient within a fluid element initially at small enough $|x| > 0$ grows inviscidly until the fluid element hits the shock.

The time at which a fluid element initially at $x$ falls into the growing shock is $t_x \approx 1/[\xi_0(a - b|x/L|^p)]$. The times of arrival at the shock determine the fractional measure of initial points $x$ such that, before a fluid element hits the shock, the gradient magnitude increases to values $\ge |\xi| \gg \xi_0$. The gradient magnitude at $x$ grows to $\approx a\xi_0|x/L|^{-p}/[b(p + 1)]$ at $t_0$ and $\approx a\xi_0|x/L|^{-p}/bp$ at $t_x$. Thus the measure of points $x$ such that the gradient

magnitude in the fluid element initially at $x$ can grow to a value that equals or exceeds $|\xi| \gg \xi_0$ within the intervals $(0, t_0)$ or $(0, t_x)$ is $\propto |\xi|^{-1/p}$. Of the initial fluid elements that achieve a value at least $|\xi| \gg \xi_0$ before hitting the shock, the fraction that does this in the preshock interval $(0, t_0)$ is $(p/p+1)^{1/p}$.

When the fluid element has achieved the value $\xi$, squeezing has decreased its measure by a factor $\propto |\xi|^{-1}$. Finally, the residence time of the fluid element in the gradient interval $d\xi$ at $\xi$ is $dt = |\xi|^{-2}d\xi$. Putting these factors together, one obtains $\bar{Q}(\xi) \propto |\xi|^{-1-2-1/p} = |\xi|^{-3-1/p}$, where $\bar{Q}(\xi)$ is the mean of $Q(\xi, t)$ over a time interval (say $2/a\xi_0$) long enough for all fluid elements that can achieve $|\xi| \gg \xi_0$ to have hit the shock.

In [5], $\alpha = 3$ was deduced under the assumption that the fractional measure of fluid elements that can achieve gradient magnitudes $\gg \xi_0$ is $O(1)$. The measure $\propto |\xi|^{-1/p}$ found instead in the present model changes the result to $\alpha = 3 + 1/p$. For all finite $p > 0$, the form of $F$ calculated from the present model is $F(\xi) \propto \xi Q(\xi)$, within the $-\alpha$ range.

Initial sawtooth profiles, where $u(x)$ consists solely of straight-line segments, correspond to $b = 0$ (or $p = \infty$) in the model. They evolve into shocks that have finite amplitude at birth and yield $\alpha = 3$. The present model thereby is consistent with the conclusion [7–9] that isolated shocks can induce $\alpha = 3$ only if they are created with finite amplitude.

The steady state produced by spatially smooth Gaussian forcing supported by wavenumbers $O(1/L)$ can be interpreted in terms of this model. Such forcing induces smooth profiles corresponding to $p = 2$ near points of extremal slope. In the absence of force, a shock forms from steepening of slope at a point of maximally negative slope. Smooth change of the velocity field due to Gaussian forcing can change the location of the point of maximally negative slope as a function of time. But such forcing does not change the nature of the shock formation phenomenon because the quadratic decrease of slope magnitude away from the point of negative maximum survives. Once the slope magnitude at negative maximum is large compared to $\xi_0$, the forcing should have no significant effect on either the progression to shock birth or the initial shock growth after birth. The value $\alpha = 7/2$ corresponding to $p = 2$ is the steady-state result.

Forcing that supports the general case $p \neq 2$ in a statistically steady state can be constructed as follows: Let the forcing consist of a set of $\delta$-functions in time, spaced at time intervals $O(1/\xi_0)$. Let each such $\delta$-function create an increment to $u(x)$ that consists of straight-line segments of length $O(L)$ with $O(u_0/L)$ positive slope, smoothly joined to interposed negative-slope regions of $O(L)$ or shorter lengths. In each negative-slope region let there be a point of maximally negative slope surrounded by a neighborhood in which $u(x)$ has the form $\xi_0(-ay+by|y/L|^p)$, where $y$ is the distance from the point of maximally negative slope. The values of $a$ and $b$ can change stochastically from one such region to another. Under these conditions, the negative-slope increment to $u(x)$ created by each $\delta$-function force field is added to an existing field that has locally constant slope with $O(1)$ probability. Therefore the points of maximally-negative slope are at the special points $y = 0$, and the consequent shock development yields $\alpha = 3 + 1/p$ for all $p > 0$, as in the initial-value case.

All cases of the model except $p = 2$, $\alpha = 7/2$ require special shapes of the velocity field prior to shock formation and, therefore, precise phase relations of spatial Fourier components. If the forcing field is spatially homogeneous and has an infinitely short coherence time (white forcing), the effective forcing is Gaussian and these shapes and phase relations cannot be realized. The white, homogeneous forcing assumed in writing the $B$ term in (2) implies $p = 2$, $\alpha = 7/2$.

At the end of [5], it was noted that the value of $\alpha$ depends on how likely it is for a shock collision to interrupt the steepening of negative gradients. It was argued that collisions should be infrequent enough that inviscid steepening should survive for most fluid elements of negative $\xi$ until $|\xi| = O(\xi_S)$ is reached. This implies $\alpha = 3$. The present shock-growth model, following the earlier ones in [6,7], says instead that fluid elements with large negative $\xi$ are inevitably close to shocks that soon swallow them if $p = O(1)$. For most fluid elements with negative $\xi$, the inviscid steepening is terminated by collision with the shocks while $|\xi|$ is still $O(\xi_0)$.

The analysis in [7] is done in terms of a split of $u$ and $\xi$ into parts exterior to shocks and parts interior to shocks, in the limit $R \to \infty$:

$$u(x, t) = u_e(x, t) + u_i(x, t), \quad \xi(x, t) = \xi_e(x, t) + \xi_i(x, t) \tag{12}$$

It is of interest to discuss how to make the split (12) when $R$ is large but not infinite, and to express the preceding analysis in terms of the split-field representation. The split into interior and exterior fields is analyzed in [8,9] by means of matched asymptotic expansions.

If the $R = \infty$ field consists of infinitely thin shocks surrounded by fluid in which $|\xi| < \xi_1$, where $\xi_1$ is some finite bound, the split is clear and unambiguous. At large finite $R$, one can isolate a small region surrounding each shock [7] as $u_i$, and consistently let the widths of these regions shrink to zero at $R = \infty$. E and Vanden Eijnden show that the result is a simple set of statistical equations relating $\xi_e$ and the shock jumps:

$$\langle \xi_e \rangle + \rho \langle s \rangle = 0, \tag{13}$$

$$F_e(\xi_e, t) = \frac{\rho}{2} \int s\,[V_-(\xi_e, s, t) + V_+(\xi_e, s, t)]\,ds. \tag{14}$$

Here $\rho$ is the number density of shocks, $s$ is shock-jump strength ($< 0$), $F_e$ is the viscous term in an equation of motion like (2) for the probability density $Q_e(\xi_e)$ of the exterior field, and $V_-$ ($V_+$) is the probability that a shock of strength $s$ has a left (right) environment with gradient $\xi_e$.

These equations have a direct physical interpretation. Equation (14) expresses the viscous term as the rate at which $Q_e(\xi_e)$ is diminished through the swallowing of fluid with gradient $\xi_e$ by shocks. The time derivative of (13) is an expression of the fact that the rate of change of shock strength is given by the product of the convergence velocity and the negative of the gradient of the external field, as the latter is swallowed by the shock.

The split into $\xi_e$ and $\xi_i$ is less clear if there is a $-\alpha$ range at $R = \infty$ that includes $|\xi|$ values larger than any finite bound. Then the $\xi_e$ field is not smooth. As $R$ approaches infinity, it is not possible to form an $R$-dependent parameter $\xi_1$ such that all field with $|\xi| < \xi_1$ belongs to $\xi_e$ and all field with $|\xi| > \xi_1$ belongs to $\xi_i$. Instead the division into interior and exterior fields must be made individually at each shock.

The equation of motion for $\xi$ obtained by differentiation of (1) contains the steepening term $\xi^2$ and the dissipation term $\nu\xi_{xx}$. One possibility for splitting the $\xi$ field is to let $\xi_e$ include all points $\xi > 0$ and all points $\xi < 0$ that are exterior to defined boundaries of the shock shoulders. A shock-shoulder boundary could be defined as a point where, as one moves through the field toward the shock, $\xi^2/|\nu\xi_{xx}|$ first falls below some prescribed ratio, say 10. Then $\xi_i$ constitutes the field interior to the left and right shock-shoulder boundaries. This gives a $\xi_e$ field that extends up to values $|\xi| = O(\xi_S)$ at some points, with a $Q_e(\xi_e)$ that must deviate from powerlaw behavior for such $|\xi|$ values. The qualitative behaviors of $\xi_e$ and $\xi_i$ are independent of the precise value prescribed for the critical ratio of steepening term to dissipation term.

Since $|\xi_e|$ extends to $\infty$ in the limit, it is not obvious that all interactions between exterior and interior fields have a form consistent with (13) and (14). In the limit, the $1/R|\xi|$ range belongs to $\xi_i$, which includes the shock shoulders. Both $\xi_e$ and $\xi_i$ contribute to the total-field probability density $Q(\xi_M)$, and this fact remains as $R \to \infty$.

It is been remarked above that $F(\xi)$ is positive in the $-1$ range, while (14) gives negative $F_e(\xi_e)$. This emphasises that the region of $\xi$ contributing to the $-1$ range in each individual shock boundary layer should be assigned to $\xi_i$.

Equations similar to (2), (5), and (8) can be written for $Q_e(\xi_e)$:

$$(Q_e)_t = \xi_e Q_e + (\xi_e^2 Q_e)_\xi + B(Q_e)_{\xi\xi} + F_e, \tag{15}$$

$$Q_e(\xi_e) \approx |\xi_e|^{-3} \int_{-\infty}^{\xi_e} \xi'_e F_e(\xi'_e) d\xi'_e \quad (-\xi_e \gg \xi_0), \tag{16}$$

$$\int_{-\xi_\alpha}^{\infty} \xi_e (Q_e)_t(\xi_e) d\xi_e = \xi_\alpha^3 Q_e(-\xi_\alpha) + \int_{-\xi_\alpha}^{\infty} \xi_e F_e(\xi_e) d\xi_e. \tag{17}$$

Although the equations look the same, $F_e$ and $F$ behave differently for negative arguments.

The parameter $\xi_M$ has no special significance here because the $-\alpha$ range of $Q_e$ is not masked by the $-1$ range. The latter belongs to $\xi_i$. Therefore $\xi_M$ has been replaced in (17) by $\xi_\alpha$, which is any value within the $-\alpha$ range that satisfies the limiting relations

$$\xi_\alpha/\xi_0 \to \infty, \quad \xi_\alpha/\xi_S \to 0 \quad (R \to \infty) \tag{18}$$

If $\alpha = 3$, (17), like (8), exhibits a boundary flow that does not vanish at $R = \infty$.

The analog of (4), $\int_{-\infty}^{\infty} \xi_e F_e(\xi_e) d\xi_e = 0$, holds in steady states. In general transient states, (13) implies that there is an additional term involving the rate of change of shock jumps.

The power-law behavior of $Q_e(\xi_e)$ must change to a faster (eventually faster-than-algebraic) fall-off for $|\xi_e| \geq O(\xi_S)$. The generic form may be written

$$Q_e(\xi_e) \approx \xi_0^{\alpha-1} |\xi_e|^{-\alpha} \tilde{Z}(\xi_e/R\xi_0) \quad (-\xi_e \gg \xi_0). \tag{19}$$

Here $\tilde{Z}$ vanishes strongly at $\xi_e = -\infty$ and is $O(1)$ at $\xi_e = -0$. The precise form of $\tilde{Z}$ is $\alpha$-dependent. In contrast to (9), there is no $-1$ range. The prefactor in (19) gives the consistency property $Q_e(-\xi_0) = O(1/\xi_0)$ if the $-\alpha$ range is extrapolated toward the central peak of $Q_e$.

Substitution of (19) into (16) yields

$$\int_{-\infty}^{-\xi_\alpha} \xi_e F_e(\xi_e) d\xi_e = (\xi_0/\xi_\alpha)^{\alpha-3} O(\xi_0^2). \tag{20}$$

If $\alpha > 3$, the dissipation measured by $\int_{-\xi_\alpha}^{\infty} \xi_e F_e(\xi_e) d\xi_e$ equals the total dissipation of $\xi_e$ in the limit $R \to \infty$. If $\alpha = 3$, (20) shows that there is additionally an essential contribution $\int_{-\infty}^{-\xi_\alpha} \xi_e F_e(\xi_e) d\xi_e$ that is $O(\xi_0^2)$ in the limit. This arises from $O(\xi_0^2/\xi_S^2)$ levels of $F_e(\xi_e)$ needed to induce the fast fall-off of $Q_e(\xi_e)$ at $|\xi_e| = O(\xi_S)$. The fast fall off occurs also if $\alpha > 3$, but in that case (20) shows that the associated contribution $\int_{-\infty}^{-\xi_\alpha} \xi_e F_e(\xi_e) d\xi_e$ is a vanishing part of the overall budget in the $R \to \infty$ limit.

In [7], the limit "$-\infty$" in (16) above and other equations is taken to mean a point within the infinite $-\alpha$ range, rather than true negative infinity ($\ll -\xi_S$). In other words "$-\infty$" is taken to mean $-\xi_\alpha$, as defined by (18) in the limit $R = \infty$. This causes no problem if $\alpha > 3$. If $\alpha = 3$, it is clear from (20) that "$-\infty$" must stay at true negative infinity.

If the possibility $\alpha = 3$ is to be examined, clearly one cannot confine attention solely to the strict $-\alpha$ powerlaw range and ignore the transition region at $|\xi_e| = O(\xi_S)$. The case $\alpha = 3$ implies that, for most fluid elements with steepening negative $\xi$, the inviscid steepening halts at $\xi$ within the transition region. The magnitude of $F_e(\xi_e)$ at $|\xi_e| = O(\xi_S)$ required by $\alpha = 3$ signifies that shocks have a relatively high concentration in fluid with such $\xi_e$. In comparison with $\alpha > 3$, shocks are moved from environments with $|\xi_e| \ll \xi_S$ to environments in the transition region $|\xi_e| = O(\xi_S)$.

If an explicit shock-growth model is not adopted to fix $F_e$, it is possible *a priori* that shocks with environments in the transition region could fall within the picture invoked by [7], in which shocks are created at zero amplitude and the balance is described fully by (13) and (14). In this case it is only needed to include the transition region of shock environments in $\langle \xi_e \rangle$ when calculating the balance (13) for $\alpha = 3$. However, it is also possible *a priori* that $F_e(\xi_e)$ is weighted to sufficiently large $|\xi_e|$ that (14) is not an accurate description of the dissipation mechanism for $\alpha = 3$, or that $F_e(\xi_e)$ includes interactions with shocks created at finite amplitude as also considered in [7].

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* Electronic address: rhk@lanl.gov

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